Introduction to Functions and Graphs

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The essence of mathematics is not to make simple things complicated, but to make complicated things simple.
—*Stanley Gudder*

Have you ever thought about how we "live by the numbers"? Money, digital televisions, speed limits, grade point averages, gas mileages, and temperatures are all based on numbers. When we are told what our weight, blood pressure, body mass index, and cholesterol level are, these numbers can even affect how we feel about ourselves. Numbers permeate our society.

People are concerned about our environment and how it is changing. Do cars and their carbon dioxide emissions contribute to global warming? Conventional cars are inherently inefficient because they burn gasoline when they are not moving. Hybrid vehicles may be a viable option, but no doubt numbers will be used to make a decision. Rates of change, consumption, efficiency, and pollution levels are all described by numbers.

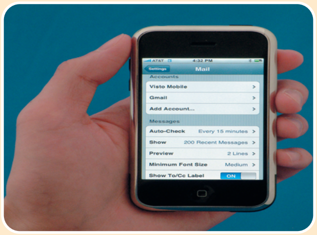
Numbers are part of mathematics, but mathematics is *much more* than numbers. Mathematics also includes techniques to analyze these numbers and to guide our decisions about the future. Mathematics is used not only in science and technology; it is also used to describe almost every facet of life, including consumer behavior and the Internet.

In this chapter we discuss numbers and how functions are used to perform computations with these numbers. Understanding numbers and mathematical concepts is essential to understanding and dealing with the many changes that will occur in our lifetimes. Mathematics makes life easier!

Source: Andrew Frank, "Plug-in Hybrid Vehicles for a Sustainable Future," *American Scientist*, March–April, 2007.

1.1 Numbers, Data, and Problem Solving

- **Recognize common sets of numbers**
- **Evaluate expressions by applying the order of operations**
- **Learn scientific notation and use it in applications**
- **Apply problem-solving strategies**



Introduction

Because society is becoming more complex and diverse, our need for mathematics is increasing dramatically each year. Numbers are essential to our everyday lives. For example, the iPhone is 4.5 inches in height, 2.4 inches in width, and 0.46 inch in thickness. It has an 8-gigabyte flash drive, a 2-megapixel camera, and 480-by-320-pixel screen resolution, and it can operate at temperatures between 32° and 95°F. (**Source:** Apple Corporation.)

Mathematics not only provides numbers to describe new products, but also gives us problem-solving strategies. This section discusses basic sets of numbers and introduces some essential problem-solving strategies.

Sets of Numbers

One important set of numbers is the set of **natural numbers**. This set comprises the *counting numbers* $N = \{1, 2, 3, 4, \ldots\}$. Natural numbers can be used when data are positive and not presented in fractional parts.

The **integers** $I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ are a set of numbers that contains the natural numbers, their additive inverses (negatives), and 0. Historically, negative numbers were not readily accepted. Today, however, when a person overdraws a personal checking account for the first time, negative numbers quickly take on meaning. There is a significant difference between a positive and a negative balance.

A **rational number** can be expressed as the *ratio* of two integers $\frac{p}{q}$, where $q \neq 0$. Rational numbers include the integers. Examples of rational numbers are

$$\frac{2}{1}, \frac{1}{3}, -\frac{1}{4}, \frac{-50}{2}, \frac{22}{7}, 0, \sqrt{25}, \text{ and } 1.2.$$

Note that 0 and 1.2 are both rational numbers. They can be represented by the fractions $\frac{0}{1}$ and $\frac{12}{10}$. Because two fractions that look different can be equivalent, rational numbers have more than one form. A rational number can always be expressed in a decimal form that either *repeats* or *terminates*. For example, $\frac{2}{3} = 0.\overline{6}$, a repeating decimal, and $\frac{1}{4} = 0.25$, a terminating decimal. The overbar indicates that $0.\overline{6} = 0.6666666\ldots$.

Real numbers can be represented by decimal numbers. Since every rational number has a decimal form, real numbers include rational numbers. However, some real numbers cannot be expressed as a ratio of two integers. These numbers are called **irrational numbers**. The numbers $\sqrt{2}$, $\sqrt{15}$, and π are examples of irrational numbers. They can be represented by nonrepeating, nonterminating decimals. Note that for any positive integer a, if $\sqrt{a}$ is not an integer, then $\sqrt{a}$ is an irrational number.

Real numbers include both rational and irrational numbers and can be *approximated* by a terminating decimal. Examples of real numbers include

$$2, -10, -131.3337, \frac{1}{3} = 0.\overline{3}, -\sqrt{5} \approx -2.2361, \text{ and } \sqrt{11} \approx 3.3166.$$

NOTE The symbol $\approx$ means **approximately equal**. This symbol is used in place of an equals sign whenever two unequal quantities are close in value. For example, $\frac{1}{4} = 0.25$, whereas $\frac{1}{3} \approx 0.3333$.

CLASS DISCUSSION

The number 0 was invented well after the natural numbers. Many societies did not have a zero—for example, there is no Roman numeral for 0. Discuss some possible reasons for this.

EXAMPLE 1 Classifying numbers

Classify each real number as one or more of the following: natural number, integer, rational number, or irrational number.

$$5, -1.2, \frac{13}{7}, -\sqrt{7}, -12, \sqrt{16}$$

SOLUTION

5: natural number, integer, and rational number

-1.2: rational number

$\frac{13}{7}$: rational number

$-\sqrt{7}$: irrational number

-12: integer and rational number

$\sqrt{16} = 4$: natural number, integer, and rational number

Now Try Exercise 7

Order of Operations

Does $6 - 3 \cdot 2$ equal 0 or 6? Does -5^2 equal 25 or -25? Figure 1.1 correctly shows that $6 - 3 \cdot 2 = 0$ and that $-5^2 = -25$. Because multiplication is performed before subtraction, $6 - 3 \cdot 2 = 0$. Similarly, because exponents are evaluated before performing negation, $-5^2 = -25$. It is essential that algebraic expressions be evaluated consistently, so the following rules have been established.

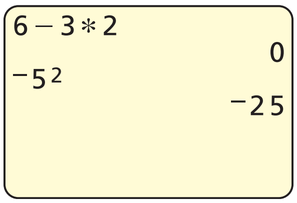


Figure 1.1

Order of Operations

Using the following order of operations, first perform all calculations within parentheses, square roots, and absolute value bars and above and below fraction bars. Then use the same order of operations to perform any remaining calculations.

1. Evaluate all exponents. Then do any negation *after* evaluating exponents.
2. Do all multiplication and division from *left to right*.
3. Do all addition and subtraction from *left to right*.

EXAMPLE 2 Evaluating arithmetic expressions

Evaluate each expression by hand.

(a) $3(1 - 5)^2 - 4^2$ **(b)** $\frac{10 - 6}{5 - 3} - 4 - |7 - 2|$

SOLUTION

(a)
$$\begin{aligned} 3(1 - 5)^2 - 4^2 &= 3(-4)^2 - 4^2 \\ &= 3(16) - 16 \\ &= 48 - 16 \\ &= 32 \end{aligned}$$

(b)
$$\begin{aligned} \frac{10 - 6}{5 - 3} - 4 - |7 - 2| &= \frac{4}{2} - 4 - |5| \\ &= 2 - 4 - 5 \\ &= -2 - 5 \\ &= -7 \end{aligned}$$

NOTE $(-4)^2 = (-4)(-4) = 16$ and $-4^2 = -(4)(4) = -16$.

Now Try Exercises 19 and 21

Scientific Notation

Numbers that are large or small in absolute value are often expressed in scientific notation. Table 1.1 lists examples of numbers in **standard (decimal) form** and in **scientific notation**.

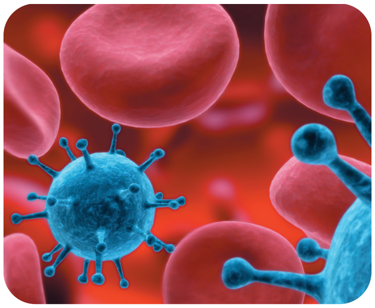


Table 1.1

Standard Form	Scientific Notation	Application
93,000,000 mi	9.3×10^7 mi	Distance to the sun
13,517	1.3517×10^4	Radio stations in 2005
9,000,000,000	9×10^9	Estimated world population in 2050
0.00000538 sec	5.38×10^{-6} sec	Time for light to travel 1 mile
0.000005 cm	5×10^{-6} cm	Size of a typical virus

To write 0.00000538 in scientific notation, start by moving the decimal point to the right of the first nonzero digit, 5, to obtain 5.38. Since the decimal point was moved six places to the *right*, the exponent of 10 is -6. Thus, $0.00000538 = 5.38 \times 10^{-6}$. When the decimal point is moved to the *left*, the exponent of 10 is positive, rather than negative. Here is a formal definition of scientific notation.

Calculator Help
To display numbers in scientific notation, see Appendix A (page AP-2).

Scientific Notation

A real number r is in **scientific notation** when r is written as $c \times 10^n$, where $1 \leq |c| < 10$ and n is an integer.

An Application Nanotechnology involves extremely small electrical circuits. Someday this technology may use the movement of the human body to power tiny devices such as pacemakers. The next example demonstrates how scientific notation appears in the description of this new technology.

EXAMPLE 3 Analyzing the energy produced by your body

Nanotechnology is a technology of the very small: on the order of one billionth of a meter. Researchers are looking to power tiny devices with energy generated by the human body. (**Source:** Z. Wang, "Self-Powered Nanotech," *Scientific American*, January 2008.)

(a) Write one billionth in scientific notation.

(b) While typing, a person's fingers generate about 2.2×10^{-3} watt of electrical energy. Write this number in standard (decimal) form.

SOLUTION

(a) One billionth can be written as $\frac{1}{1{,}000{,}000{,}000} = \frac{1}{10^9} = 1 \times 10^{-9}$.

(b) Move the decimal point in 2.2 three places to the left: $2.2 \times 10^{-3} = 0.0022$.

Now Try Exercise 83

The next two examples illustrate how to evaluate expressions involving scientific notation.

EXAMPLE 4 Evaluating expressions by hand

Evaluate each expression. Write your result in scientific notation and standard form.

(a) $(3 \times 10^3)(2 \times 10^4)$ **(b)** $(5 \times 10^{-3})(6 \times 10^5)$ **(c)** $\dfrac{4.6 \times 10^{-1}}{2 \times 10^2}$

SOLUTION

(a)

$$\begin{aligned}(3 \times 10^3)(2 \times 10^4) &= 3 \times 2 \times 10^3 \times 10^4 && \text{Commutative property}\\ &= 6 \times 10^{3+4} && \text{Add exponents.}\\ &= 6 \times 10^7 && \text{Scientific notation}\\ &= 60{,}000{,}000 && \text{Standard form}\end{aligned}$$

(b)

$$\begin{aligned}(5 \times 10^{-3})(6 \times 10^5) &= 5 \times 6 \times 10^{-3} \times 10^5 && \text{Commutative property}\\ &= 30 \times 10^2 && \text{Add exponents.}\\ &= 3 \times 10^3 && \text{Scientific notation}\\ &= 3000 && \text{Standard form}\end{aligned}$$

(c)

$$\begin{aligned}\frac{4.6 \times 10^{-1}}{2 \times 10^2} &= \frac{4.6}{2} \times \frac{10^{-1}}{10^2} && \text{Multiplication of fractions}\\ &= 2.3 \times 10^{-1-2} && \text{Subtract exponents.}\\ &= 2.3 \times 10^{-3} && \text{Scientific notation}\\ &= 0.0023 && \text{Standard form}\end{aligned}$$

Now Try Exercises 53, 55, and 57 ◀

Algebra Review
To review exponents, see Chapter R (page R-7).

Calculators Calculators often use E to express powers of 10. For example, 4.2×10^{-3} might be displayed as 4.2E–3. On some calculators, numbers can be entered in scientific notation with the [EE] key, which you can find by pressing [2nd] [,].

EXAMPLE 5 Computing in scientific notation with a calculator

Approximate each expression. Write your answer in scientific notation.

(a) $\left(\dfrac{6 \times 10^3}{4 \times 10^6}\right)(1.2 \times 10^2)$ **(b)** $\sqrt{4500\pi}\left(\dfrac{103 + 450}{0.233}\right)^3$

SOLUTION

(a) The given expression is entered in two ways in Figure 1.2. Note that in both cases

$$\left(\frac{6 \times 10^3}{4 \times 10^6}\right)(1.2 \times 10^2) = 0.18 = 1.8 \times 10^{-1}.$$

(b) Be sure to insert parentheses around 4500π and around the numerator, $103 + 450$, in the ratio. From Figure 1.3 we can see that the result is approximately 1.59×10^{12}.

Calculator Help
To enter numbers in scientific notation, see Appendix A (page AP-2).

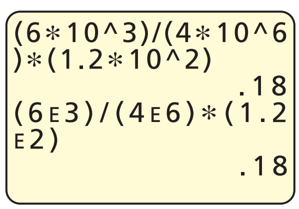


Figure 1.2

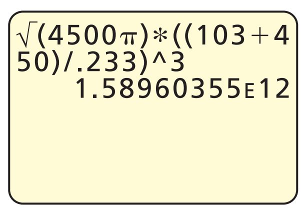


Figure 1.3

Now Try Exercises 61 and 63 ◀

EXAMPLE 6 Computing with a calculator

Use a calculator to evaluate each expression. Round answers to the nearest thousandth.

(a) $\sqrt[3]{131}$ **(b)** $\pi^3 + 1.2^2$ **(c)** $\dfrac{1 + \sqrt{2}}{3.7 + 9.8}$ **(d)** $|\sqrt{3} - 6|$

SOLUTION

(a) On some calculators the cube root can be found by using the MATH menu. If your calculator does not have a cube root key, enter 131^(1/3). From the first two lines in Figure 1.4, we see that $\sqrt[3]{131} \approx 5.079$.

(b) Do *not* use 3.14 for the value of π. Instead, use the built-in key to obtain a more accurate value of π. From the bottom two lines in Figure 1.4, $\pi^3 + 1.2^2 \approx 32.446$.

(c) When evaluating this expression be sure to include parentheses around the numerator and around the denominator. Most calculators have a special square root key that can be used to evaluate $\sqrt{2}$. From the first three lines in Figure 1.5, $\frac{1 + \sqrt{2}}{3.7 + 9.8} \approx 0.179$.

(d) The absolute value can be found on some calculators by using the MATH NUM menus. From the bottom two lines in Figure 1.5, $|\sqrt{3} - 6| \approx 4.268$.

Algebra Review
To review cube roots, see Chapter R (page R-40).

Calculator Help
To enter expressions such as $\sqrt[3]{131}$, $\sqrt{2}$, π, and $|\sqrt{3} - 6|$, see Appendix A (page AP-2).

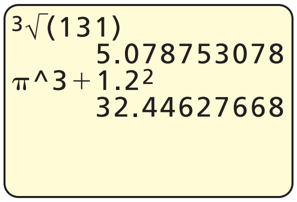


Figure 1.4

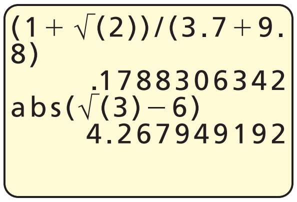


Figure 1.5

Now Try Exercises 67, 69, 71, and 73 ◀

Problem Solving

Many problem-solving strategies are used in algebra. However, in this subsection we focus on two important strategies that are used frequently: making a sketch and applying one or more formulas. These strategies are illustrated in the next three examples.

EXAMPLE 7 Finding the speed of Earth

Earth travels around the sun in an approximately circular orbit with an average radius of 93 million miles. If Earth takes 1 year, or about 365 days, to complete one orbit, estimate the orbital speed of Earth in miles per hour.

SOLUTION

Getting Started Speed S equals distance D divided by time T, $S = \frac{D}{T}$. We need to find the number of miles Earth travels in 1 year and then divide it by the number of hours in 1 year. ▶

Geometry Review
To find the circumference of a circle, see Chapter R (page R-2).

Distance Traveled A sketch of Earth orbiting the sun is shown in Figure 1.6. In 1 year Earth travels the circumference of a circle with a radius of 93 million miles. The circumference of a circle is $2\pi r$, where r is the radius, so the distance D is

$$D = 2\pi r = 2\pi(93{,}000{,}000) \approx 584{,}300{,}000 \text{ miles.}$$

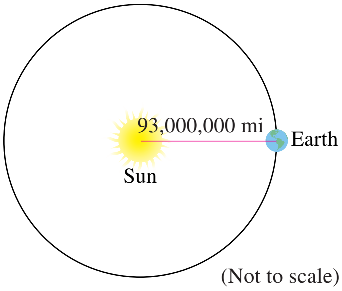


Figure 1.6 Earth's Orbit

Hours in 1 Year The number of hours H in 1 year, or 365 days, equals

$$H = 365 \times 24 = 8760 \text{ hours.}$$

Speed of Earth $S = \frac{D}{H} = \frac{584{,}300{,}000}{8760} \approx 66{,}700$ miles per hour.

Now Try Exercise 85

Many times in geometry we evaluate formulas to determine quantities, such as perimeter, area, and volume. In the next example we use a formula to determine the number of fluid ounces in a soda can.

EXAMPLE 8 Finding the volume of a soda can

The volume V of the cylindrical soda can in Figure 1.7 is given by $V = \pi r^2 h$, where r is its radius and h is its height.

(a) If $r = 1.4$ inches and $h = 5$ inches, find the volume of the can in cubic inches.

(b) Could this can hold 16 fluid ounces? (*Hint:* 1 cubic inch equals 0.55 fluid ounce.)

Figure 1.7 A Soda Can

SOLUTION

(a) $V = \pi r^2 h = \pi(1.4)^2(5) = 9.8\pi \approx 30.8$ cubic inches.

(b) To find the number of fluid ounces, multiply the number of cubic inches by 0.55.

$$30.8 \times 0.55 = 16.94$$

Yes, the can could hold 16 fluid ounces. ***Now Try Exercise 93***

Measuring the thickness of a very thin layer of material can be difficult to do directly. For example, it would be difficult to measure the thickness of a sheet of aluminum foil or a coat of paint with a ruler. However, it can be done indirectly using the following formula.

$$\text{Thickness} = \frac{\text{Volume}}{\text{Area}}$$

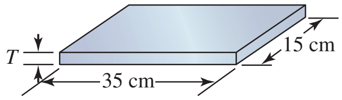
That is, the thickness of a thin layer equals the volume of the substance divided by the area that it covers. For example, if a volume of 1 cubic inch of paint is spread over an area of 100 square inches, then the thickness of the paint equals $\frac{1}{100}$ inch. This formula is illustrated in the next example.

EXAMPLE 9 Calculating the thickness of aluminum foil

A rectangular sheet of aluminum foil is 15 centimeters by 35 centimeters and weighs 5.4 grams. If 1 cubic centimeter of aluminum weighs 2.7 grams, find the thickness of the aluminum foil. (**Source:** U. Haber-Schaim, *Introductory Physical Science.*)

SOLUTION

Getting Started Start by making a sketch of a rectangular sheet of aluminum, as shown in Figure 1.8. To complete this problem we need to find the volume V of the aluminum foil and its area A. Then we can determine the thickness T by using the formula $T = \frac{V}{A}$. ▶



(Not to scale)

Figure 1.8 Aluminum Foil

NOTE For the rectangular box shape shown in Figure 1.8 on the previous page,

$$\textbf{Volume} = \underbrace{\text{Length} \times \text{Width}}_{\textbf{Area}} \times \textbf{Thickness}.$$

It follows that **Thickness** $= \frac{\textbf{Volume}}{\textbf{Area}}$.

Geometry Review
To find the area of a rectangle, see Chapter R (page R-1). To find the volume of a box, see Chapter R (page R-3).

Volume Because the aluminum foil weighs 5.4 grams and each 2.7 grams equals 1 cubic centimeter, the volume of the aluminum foil is

$$\frac{5.4}{2.7} = 2 \text{ cubic centimeters.}$$ Divide weight by density.

Area The aluminum foil is rectangular with an area of $15 \times 35 = 525$ square centimeters.

Thickness The thickness of **2** cubic centimeters of aluminum foil with an area of **525** square centimeters is

$$\textbf{Thickness} = \frac{\textbf{Volume}}{\textbf{Area}} = \frac{\mathbf{2}}{\mathbf{525}} \approx 0.0038 \text{ centimeter.}$$

Now Try Exercise 89

1.1 Putting It All Together

Numbers play a central role in our society. Without numbers, data could be described qualitatively but not quantitatively. For example, we could say that the day seems hot but would not be able to give an actual number for the temperature. Problem-solving strategies are used in almost every facet of our lives, providing the procedures needed to systematically complete tasks and perform computations.

The following table summarizes some of the concepts in this section.

Concept	Comments	Examples
Natural numbers	Sometimes referred to as the *counting numbers*	$1, 2, 3, 4, 5, \ldots$
Integers	Include the natural numbers, their opposites, and 0	$\ldots, -2, -1, 0, 1, 2, \ldots$
Rational numbers	Include integers; all fractions $\frac{p}{q}$, where p and q are integers with $q \neq 0$; all repeating and all terminating decimals	$\frac{1}{2}, -3, \frac{128}{6}, -0.335, 0,$ $0.25 = \frac{1}{4}, 0.\overline{33} = \frac{1}{3}$
Irrational numbers	Can be written as nonrepeating, nonterminating decimals; cannot be a rational number; if a square root of a positive integer is not an integer, it is an irrational number.	$\pi, \sqrt{2}, -\sqrt{5}, \sqrt[3]{7}, \pi^4$

Concept	Comments	Examples
Real numbers	Any number that can be expressed in standard (decimal) form Include the rational numbers and irrational numbers	$\pi, \sqrt{7}, -\frac{4}{7}, 0, -10, 1.237$ $0.\overline{6} = \frac{2}{3}, 1000, \sqrt{15}, -\sqrt{5}$
Order of operations	Using the following order of operations, first perform all calculations within parentheses, square roots, and absolute value bars and above and below fraction bars. Then perform any remaining calculations. 1. Evaluate all exponents. Then do any negation *after* evaluating exponents. 2. Do all multiplication and division from *left to right.* 3. Do all addition and subtraction from *left to right.*	$-4^2 - 12 \div 2 - 2 = -16 - 12 \div 2 - 2$ $= -16 - 6 - 2$ $= -22 - 2$ $= -24$ $\frac{2 + 4^2}{3 - 3 \cdot 5} = \frac{2 + 16}{3 - 15}$ $= \frac{18}{-12}$ $= -\frac{3}{2}$
Scientific notation	A number in the form $c \times 10^n$, where $1 \leq \lvert c \rvert < 10$ and n is an integer Used to represent numbers that are large or small in absolute value	$3.12 \times 10^4 = 31{,}200$ $-1.4521 \times 10^{-2} = -0.014521$ $5 \times 10^9 = 5{,}000{,}000{,}000$ $1.5987 \times 10^{-6} = 0.0000015987$

1.1

Classifying Numbers

Exercises 1–6: Classify the number as one or more of the following: natural number, integer, rational number, or real number.

1. $\frac{21}{24}$ (Fraction of people in the United States completing at least 4 years of high school)

2. 20,082 (Average cost in dollars of tuition and fees at a private college in 2004)

3. 7.5 (Average number of gallons of water used each minute while taking a shower)

4. 25.8 (Nielsen rating of the TV show *Grey's Anatomy* the week of February 12–18, 2007)

5. $90\sqrt{2}$ (Distance in feet from home plate to second base on a baseball field)

6. -71 (Wind chill when the temperature is -30°F and the wind speed is 40 mph)

Exercises 7–10: Classify each number as one or more of the following: natural number, integer, rational number, or irrational number.

7. $\pi, -3, \frac{2}{9}, \sqrt{9}, 1.\overline{3}, -\sqrt{2}$

8. $\frac{3}{1}, -\frac{5}{8}, \sqrt{7}, 0.\overline{45}, 0, 5.6 \times 10^3$

9. $\sqrt{13}, \frac{1}{3}, 5.1 \times 10^{-6}, -2.33, 0.\overline{7}, -\sqrt{4}$

10. $-103, \frac{21}{25}, \sqrt{100}, -\frac{5.7}{10}, \frac{2}{9}, -1.457, \sqrt{3}$

Exercises 11–16: For the measured quantity, state the set of numbers that most appropriately describes it. Choose from the natural numbers, integers, and rational numbers. Explain your answer.

11. Shoe sizes
12. Populations of states
13. Gallons of gasoline
14. Speed limits